joint was considered to move at a constant velocity from the initial position to the $\hat{e}_3$ axis in 300 s. The cost function matrix B is the 5×5 identity matrix. The initial value of the joint position is approximately 0.598 m, h_e is 0.62 m, and L_1 is 1.75 m.

The NLP routine progressed smoothly to the optimal solution even though the initial guess of the state and control histories gave no information about the optimal solution. The control histories and optimal cost varied with the number of segments used. The use of 6, 12, 18, and 24 segments resulted in optimal costs of 2.941, 2.301, 2.317, and 2.320, respectively. When 12 or more segments are used, the cost compares well to the previous result of 2.335.[1] The results presented here are for the 12-segment formulation that required 30 s of execution time.

The solutions for the external torques T_1, T_2, and T_3, shown in Figs. 3, compare very well with those of Ref. 1. Webber[3] showed that changes in the joint acceleration profile may significantly affect the control histories and optimal cost. In the previous case, the motion of the joint is prescribed. When joint acceleration $\ddot{J}_r$ is instead included as an additional control variable and appropriately weighted in the cost function ($B_{66}=2\cdot 10^6$, leaving the weights on the other controls unchanged), the cost is significantly reduced to 0.726. As a third case, since thrusters normally have a fixed output, we have also examined the case in which the external torque controls are restricted to be constant throughout the detumbling process. Then, the optimal external torque magnitudes are found. For this case, the cost is 0.767, slightly larger, as expected, than the case in which the external torques are unconstrained. Finally, since the spin rate of the OMV, ω_3, is unspecified at the final time, and the attitude of the OMV is unspecified at the final time, we have solved a case in which the external controls are eliminated. This forces the internal controls to perform the entire work of detumbling the disabled satellite but would eliminate the need for fuel to be used during the detumbling process. The optimal cost for this case is 0.828. Space does not permit us to show the motion and control time histories for all of these cases, but they are available.[10] One observation that is true for all of the cases described is that the detumbling process is very benign; no large angular accelerations are produced. In addition, the required external torque magnitudes are very modest, on the order of 0.01-0.001 N-m.

Conclusions

The method of direct collocation and nonlinear programming presented here works well when applied to the optimal control problem of satellite attitude control. The formulation is straightforward and produces good results in a relatively small amount of time on a Cray X/MP with no a priori information about the optimal solution. The addition of the joint acceleration to the controls significantly reduces the control magnitudes and optimal cost. The restrictions of constant external torques or zero external torque have an insignificant effect on the state and internal control histories and result in only a small increase in the optimal cost. In all cases, the torques and accelerations are modest and the optimal cost is very modest.

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Equivalence of Kane's, Gibbs-Appell's, and Lagrange's Equations

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Introduction

In a recent Note,[1] Kane's generalized forces and equations were derived from a first principle—the work-energy form of Newton's second law. Lagrange's equations were also derived from this form; although it differs conceptually from the usual virtual work (d'Alembert principle) derivations, the steps are similar. These parallel derivations showed the commonality of Kane's and Lagrange's equations and the automatic occurrence of the virtual work terms, which are identified as geometric work. In this Note, the Gibbs-Appell equations are derived via a modification of the derivation of Kane's equations, and the equivalence of these three forms is derived for both the holonomic and nonholonomic cases.

The common feature of these equations is the transformation to generalized coordinates such that system constraint forces can be eliminated—a great convenience. In the usual derivations, this is attributed to the properties of virtual displacements[5-13] or other displacements that are chosen properly.[2,11,14] In all cases, explicit time-varying (rheonomic) kinematrical terms are discarded, often without comment, or with explanations that are convoluted and unenlightening.[2,5-13,15,16] The present derivations entail no such assumptions or explanations in establishing these results.

Derivations

The derivations are for a system of p constant-mass particles. Extension to rigid bodies is straightforward. Newton's second law for mass m_i, located at $\boldsymbol{r}^i$ in an inertial reference frame is

$$\boldsymbol{F}_i = m_i\ddot{\boldsymbol{r}}^i = m_i\boldsymbol{a}^i, \qquad i = 1,p \tag{1}$$

Here, $\boldsymbol{F}_i$ is the sum of all forces acting on the particle: the known applied and external forces and the constraint forces. The total work of all forces on the p particles undergoing some possible displacements $\mathrm{d}\boldsymbol{r}^i$, $i=1,p$, is

$$\mathrm{d}W = \sum_{i=1}^{p}\boldsymbol{F}_i\cdot\mathrm{d}\boldsymbol{r}^i = \sum_{i=i}^{p}m_i\ddot{\boldsymbol{r}}^i\cdot\mathrm{d}\boldsymbol{r}^i = \sum_{i=1}^{p}m_i\boldsymbol{a}^i\cdot\mathrm{d}\boldsymbol{r}^i \tag{2}$$

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To incorporate the geometric (holonomic) constraints, the $3p$ coordinates r^i are transformed to n generalized coordinates q_j, viz.,

$$r^i = r^i(q_1, q_2, \ldots, q_n, t), \qquad i = 1, p \tag{3}$$

The usual and most important cases are where $n \ll 3p$. The possible differential displacements and absolute velocities of the m_i are

$$\mathrm{d}r^i = \sum_{j=1}^{n} \frac{\partial r^i}{\partial q_j} \mathrm{d}q_j + \frac{\partial r^i}{\partial t} \mathrm{d}t, \qquad i = 1, p \tag{4}$$

$$v^i = \dot{r}^i = \sum_{j=1}^{n} \frac{\partial r^i}{\partial q_j} \dot{q}_j + \frac{\partial r^i}{\partial t}, \qquad i = 1, p \tag{5}$$

The quantities in Eqs. (4) and (5) are ordinary differentials and derivatives, occurring with the passage of time $\mathrm{d}t$.

It is useful to define another n variables u_r, which are linear functions of the n $\dot{q}_j$ per the reciprocal relations

$$u_r = \sum_{j=1}^{n} Y_{rj}\dot{q}_j + Z_r, \qquad r = 1, n \tag{6}$$

$$\dot{q}_j = \sum_{r=1}^{n} W_{jr} u_r + X_j, \qquad j = 1, n \tag{7}$$

The u_r are often called generalized speeds (also nonholonomic velocities, quasivelocities, etc.) and are not unique.

Nonholonomic constraints are linear relations among either the u_r, or the $\dot{q}_j$; for example, for m nonholonomic constraints:

$$u_s = \sum_{r=1}^{n-m} A_{sr} u_r + B_s, \qquad s = n - m + 1, n \tag{8}$$

Substitution of Eq. (8) into Eq. (7) yields a more general expression for the $\dot{q}_j$,

$$\dot{q}_j = \sum_{r=1}^{k} W_{jr} u_r + X_j, \qquad j = 1, n \tag{9}$$

In Eqs. (6-9), the Y_{rj} and Z_r, W_{jr} and X_j, and A_{sr} and B_s may be functions of the q_j and t. Equations (7) and (9) differ in the limit on the sum, n or k, respectively. In Eq. (9), k may be $n - m$ or n, depending on whether the nonholonomic constraints are incorporated. In the former case, the k u_r are a nonholonomic subset of the n generalized speeds of Eq. (6). This notation provides flexibility in the derivations. If the limit is n, the nonholonomic constraints cannot be included, and Eqs. (9) and (7) are the same. The limit k denotes the optional case in which the nonholonomic constraints may or may not be incorporated.

Kane's Equations

To get the particle velocities in the necessary form, Eqs. (9) are substituted into Eqs. (5), and terms are collected:

$$v^i = \sum_{r=1}^{k} v_r^i u_r + v_t^i, \qquad i = 1, p \tag{10}$$

Each vector v_r^i is the rth partial velocity of particle i, since

$$v_r^i = \frac{\partial v^i}{\partial u_r}, \qquad r = 1, k \tag{11}$$

Absolute angular velocities would also be written similarly.

The vectors v_r^i and v_t^i are, in general, functions of the q_j and t; they normally are determined by inspection after performing the indicated substitutions. The v_t^i are generated by rheonomic constraints and explicit time-varying inputs and are not coefficients of a generalized speed; they are discarded in forming Kane's equations, usually without any explanation.[2-4] (They correspond to terms that are discarded for virtual motions.)

In Eq. (2), let $\mathrm{d}r^i = v^i \mathrm{d}t$ and substitute Eq. (10). With rearrangement and collection of terms, Eq. (2) can be written as

$$\left(\sum_{r=1}^{k} \left[\sum_{i=1}^{p} v_r^i \cdot F_i + \sum_{i=1}^{p} v_r^i \cdot (-m_i a^i) \right] u_r \right) \mathrm{d}t + \left(\sum_{i=1}^{p} v_t^i \cdot (F_i - m_i a^i) \right) \mathrm{d}t = 0 \tag{12}$$

Equation (12) has two groupings of terms: the first reflects system geometry (u_r $\mathrm{d}t$) and the second reflects time ($\mathrm{d}t$). The second sum is clearly zero and provides no new information, per Eq. (1). In the first sum, the u_r and $\mathrm{d}t$ are nonzero and independent, and so the coefficient of each of these must be zero. These coefficients are comprised of the generalized active forces and generalized inertia forces, respectively defined as

$$Q_r = \sum_{i=1}^{p} v_r^i \cdot F_i, \qquad r = 1, k \tag{13}$$

$$Q_r^* = \sum_{i=1}^{p} v_r^i \cdot (-m_i a^i), \qquad r = 1, k \tag{14}$$

whence

$$Q_r + Q_r^* = 0, \qquad r = 1, k \tag{15}$$

Equations (13-15) comprise Kane's equations; they are based on Newton's second law and system geometric properties. In Eqs. (13), certain forces in the F_i will not appear in the Q_r. These include the usual so-called nonworking (or ideal) forces of constraint and, per Eqs. (2) and (12), forces normal to or with no component in the feasible directions of motion. These are forces that do no geometric work—a descriptor preferred here over virtual work since the forces are prohibited from doing work by the constraints. Such forces must be included in F_i for Newton's equation. However, in forming Eq. (13), whether or not such forces are included in the analysis (in F_i), they will not appear in the generalized active forces. Any valid definition of generalized speeds per Eqs. (6-10) yields these results.

This derivation applies to both the holonomic and nonholonomic cases. For the former, $k = n$ and Eq. (7) is used in forming the partial velocities. For the latter, $k = n - m$ and Eqs. (8) and (9) define the nonholonomic subset of generalized speeds.

Gibbs-Appell's Equations

The relationship between Gibbs-Appell's and Kane's equations has been noted[2,16,17] and debated.[17,18] Most derivations of the Gibbs-Appell equations use virtual motions[6,11-15] (or pseudoaccelerations[12]), but the Gibbs-Appell equations can be derived immediately from the foregoing approach. Differentiation of Eq. (10) gives

$$\dot{v}^i = a^i = \sum_{r=1}^{k} v_r^i \dot{u}_r + a_t^i, \qquad i = 1, p \tag{16}$$

where a_t^i contains all of the remaining time derivatives; these are functions of the q_j, u_r (or $\dot{q}_j$), and v_t^i. Comparison of Eq. (16) with Eqs. (10) and (11) shows that each $v_r^i = \partial a^i / \partial \dot{u}_r$. Hence, the generalized inertia force in Eqs. (12) and (14) can be written as

$$Q_r^* = \sum_{i=1}^{p} (m_i a^i) \cdot \frac{\partial a^i}{\partial \dot{u}_r} = \frac{\partial}{\partial \dot{u}_r} \sum_{i=1}^{p} \frac{1}{2} m_i a^i \cdot a^i = \frac{\partial}{\partial \dot{u}_r} G \tag{17}$$

where G is the Gibbs function. Since nothing else is changed in the previous derivation, the new result is

$$\frac{\partial G}{\partial \dot{u}_r} = Q_r, \qquad r = 1, k \tag{18}$$

The Q_r are given by Eq. (13). Equations (17) and (18) define the Gibbs-Appell equations in terms of generalized speeds and apply to both the nonholonomic and holonomic cases. Clearly, the Gibbs-Appell forces have the same properties as Kane's generalized forces; the forces are identical for the same generalized speeds. As before, $k = n$ or $k = n - m$ depending on the situation.

Lagrange's Equations

It was shown that Lagrange's equations can be derived from work energy as easily as from the virtual work principle[1]; Eqs. (4) are used in Eq. (2), the summations are reordered, and after some manipulations,[1,5] one obtains for the nonholonomic case:

$$\sum_{j=1}^{n}\left[\hat{Q}_j - \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right)\right] dq_j + \sum_{i=1}^{p}\left[(\boldsymbol{F}_i - m_i \ddot{\boldsymbol{r}}^i)\cdot\frac{\partial \boldsymbol{r}^i}{\partial t}\right] dt = 0 \tag{19}$$

In Eq. (19), the Lagrange generalized force is

$$\hat{Q}_j = \sum_{i=1}^{p} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}^i}{\partial q_j}, \qquad j = 1, n \tag{20}$$

and

$$T = \frac{1}{2}\sum_{i=1}^{p} m_i(\dot{\boldsymbol{r}}^i \cdot \dot{\boldsymbol{r}}^i)$$

is the system kinetic energy.

All of the differentials in Eq. (19) are independent and nonzero. The right-hand sum is from retention of the rheonomic terms in Eq. (4); every term is zero by Eq. (1). In the left-hand sum, the coefficient of each dq_j is equated to zero to yield n Lagrange's equations corresponding to the n generalized coordinates.

The Lagrange generalized forces in Eq. (20) have the same properties as Kane's and Gibbs-Appell's generalized forces: forces that do no geometric work are eliminated and, consequently, may or may not be included in forming the $\boldsymbol{F}_i$. The respective generalized forces are identical if the $u_r = \dot{q}_j$, $j = r = 1, n$, whence $\boldsymbol{v}_r^i = \partial \boldsymbol{r}^i/\partial q_r$.

The operations in deriving Lagrange's equations make it cumbersome to use generalized speeds per Eqs. (6) and (7) (although it is possible[9]) and prohibit the direct incorporation of nonholonomic constraints as in Kane's and Gibbs-Appell's equations. Nonholonomic constraints are usually treated by appending them to T (or the Lagrangian) with Lagrange multipliers; the n Lagrange equations and m constraints are then solved conjointly in the $n + m$ generalized coordinates and multipliers.

Alternately, for the nonholonomic case, let $dq_j = \dot{q}_j\, dt$ in Eq. (19), then use Eq. (9) to transform from the $\dot{q}_j$ to the nonholonomic variables, giving

$$\sum_{j=1}^{n}\left[\hat{Q}_j - \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right)\right]\left(\sum_{r=1}^{k} W_{jr}u_r + X_j\right) dt + [0]\, dt = 0 \tag{21}$$

(The [0] denotes the rheonomic terms, which are trivially zero.) Superficially, this may not look like progress, but on expansion and reordering the summations some practical results occur:

$$\sum_{r=1}^{k}\left[\sum_{j=1}^{n}\hat{Q}_j W_{jr} - \sum_{j=1}^{n}\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right)W_{jr}\right]u_r\, dt + \sum_{j=1}^{n}\left[\hat{Q}_j - \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right)\right]X_j\, dt = 0 \tag{22}$$

As before, the u_r and dt are independent and nonzero, and so the individual coefficients must be zero. The second sum in Eq. (22) returns the previous Lagrange's equations. In the first sum, the u_r may be either holonomic or nonholonomic generalized speeds. The resulting forces are clearly identical to Kane's, Eqs. (13) and (14), and Gibbs-Appell's, Eq. (18), i.e.,

$$Q_r = \sum_{i=1}^{p} \boldsymbol{v}_r^i \cdot \boldsymbol{F}_i = \sum_{j=1}^{n} \hat{Q}_j W_{jr}, \qquad r = 1, k \tag{23}$$

$$Q_r^* = \sum_{i=1}^{p} \boldsymbol{v}_r^i \cdot (-m_i \boldsymbol{a}^i) = -\sum_{j=1}^{n}\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right)W_{jr}, \qquad r = 1, k \tag{24}$$

The utility of these results is that the left calculation in Eq. (23) may be easier for the generalized forces, yet the inertial forces are more easily computed from the kinetic energy and the nonholonomic generalized speeds on the right side of Eq. (24).

Conclusions

Kane's, Gibbs-Appell's, and Lagrange's equations can be derived from the work-energy form of Newton's second law—considering all of the forces acting on the system and total differentials—for both holonomic and nonholonomic cases. The equations come from the geometric work component of the total work. The forms are clearly equivalent for the same generalized speeds. Kane's equations can be viewed as a precursor form to Gibbs-Appell's and Lagrange's equations, in that Kane's equations would occur at an intermediate point of the other derivations. The projection equations[1,19] are also equivalent to Kane's equations.

The transformations in Eqs. (3), and subsequently Eqs. (4-9), essentially segregate dynamics problems into local and universal problems; the former are solved by Kane's, Gibbs-Appell's, and Lagrange's equations as a consequence of the transformation to generalized, or local, coordinates. The latter are solved in terms of Newtonian or relativistic universal coordinates and incorporate the discarded rheonomic terms. It is well known that a correct local solution requires a suitable reference frame in which universal effects are negligible.

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Football as a Differential Game

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Introduction

AMERICAN football provides an analogy to the aerial defense problem. The performance criterion in football is the distance upfield moved by a specific offensive player carrying the football, while the aerial defense problem instead involves a number of offensive players, and a "field" that is only roughly rectangular. A more accurate analogy might replace the goal line with a number of "goal points." But in both problems, the evader seeks to maximize the downrange distance covered before being tackled, or "intercepted," by a pursuer. In football, the evader's teammates are of interest to the pursuers because one of them may become the evader if the ball is passed to him by the initial evader. This feature is also evidently absent from the equivalent aerial defense problem.

The defensive pursuer team wants to minimize this upfield yardage by tackling the evader with the football. Tackling is modeled here as a range constraint; if the range from this evader to any pursuer falls below the *capture* range, the evader has been tackled and the play ends. But when a pass by the ball carrier is feasible, the pursuers must consider all evaders as potential receivers.

The control of any player is the direction in which he runs. Each team has a unique optimal tactic for most geometric configurations. Midfield (mirror) symmetry obviously permits a right-left choice of tactics for the evader. Multiple tactics occur more generally at *dispersal* points,[1] but in nonsymmetric configurations they are less apparent. The evader speeds are treated here as equal to or greater than the pursuer speeds.

In this game, tactics depend on both relative and real geometry of the players. This effectively doubles the order of the problem. Every player on the two teams has coordinates (x, y) so American football is a system of order $2 \times 2 \times 11 = 44$, if speeds are constant and turns are immediate. A complete solution for the game is not given here (!), but three idealized subcases are solved. These are as follows:

1) The one-on-one equal-speed problem. The question answered is: How should evader E run to maximize the distance upfield, and how should defending pursuer P run to minimize this distance?

2) The three-on-three equal-speed problem. Tactics of the players are required as functions of six pairs of (x,y) coordinates. This version permits the ball to be passed from E_1 to either E_2 or E_3, and the pursuers must defend against both possibilities.

3) The one-on-one problem when E is faster than P. Now E has the prospect of running "around" P, and curved paths result. When P is faster, optimal paths are straight.

One-on-One Tactics, Equal Speeds

The geometry at any time is shown in Fig. 1. For any positions (x_p, y_p) and (x_e, y_e), there exists a hyperbolic locus passing between the players such that the time needed for E to arrive at any point on the locus equals the time required by P to arrive within the tackle range L of the same point. The point toward which they should run is that which is farthest upfield. This point is (x_f, y_f), and whether x_f is zero or positive depends on x_e, $\Delta x = x_p - x_e$, $\Delta y = y_p - y_e$, and the tackle range L.

The two cases are shown in Fig. 2. A midfield tackle is optimal if

$$\Delta y > 0$$

$$\Delta x < L$$

$$\delta x < x_e \tag{1}$$

where δx is the x distance from E to the apex of the hyperbola,

$$\delta x = (\Delta x/2)(\Delta y/D - 1)$$

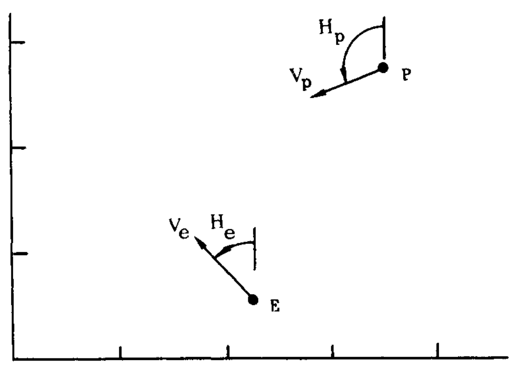

Fig. 1 General one-on-one kinematics.

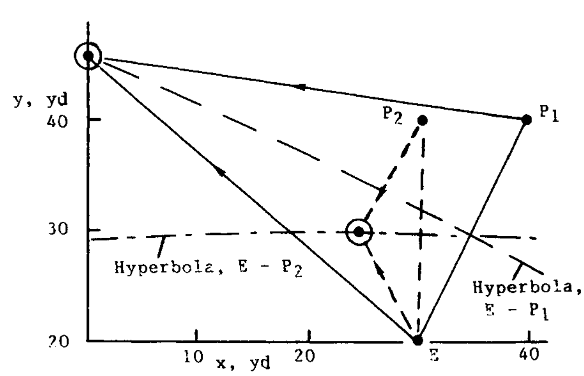

Fig. 2 Hyperbolic loci in the equal-speed case.

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